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Alternative dispersionless limit of N=2 supersymmetric KdV-type hierarchies

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Abstract

We present a systematic procedure for obtaining the dispersionless limit of a class of $N = 1$ supersymmetric systems starting from the Lax description of their dispersive counterparts. This is achieved by starting with an $N = 2$ supersymmetric system and scaling the fields in an alternative manner so as to maintain $N = 1$ supersymmetry. We illustrate our method by working out explicitly the examples of the dispersionless supersymmetric two boson hierarchy and the dispersionless supersymmetric Boussinesq hierarchy.

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1 Introduction

Dispersionless integrable systems [1] have been studied, in recent years, from various points of view. While the properties of such bosonic models are quite well understood, much remains to be learnt about their supersymmetric counterparts. For example, the Lax description of only a handful of supersymmetric dispersionless systems have been constructed, to date, by brute force [2, 3] and there is no systematic procedure for obtaining them starting from the corresponding supersymmetric dispersive Lax descriptions [4, 5, 6, 7]. Similarly, supersymmetric dispersionless systems often have more conserved charges than their dispersive counterparts and we do not yet know how to relate the new charges to the Lax function itself.

In this letter, we take a modest step and show how one can obtain, systematically, a Lax description for a select class of supersymmetric dispersionless systems starting from their dispersive counterparts. We describe in section **2** the basic procedure for taking the dispersionless limit in the Lax description itself, within the context of bosonic models. We work out some known examples to illustrate the procedure and present the Lax description of some new bosonic models. In section **3**, we extend this method to a class of supersymmetric models. We work out explicitly the example of dispersionless supersymmetric two boson hierarchy starting from the $N = 2$ supersymmetric KdV hierarchy. In section **4**, we extend the analysis and discuss the dispersionless supersymmetric Boussinesq hierarchy and present a brief conclusion in section **5**.

2 Bosonic dispersionless systems

In this section, we describe the basic procedure of taking the dispersionless limit of bosonic systems within the framework of the Lax description itself. Let us consider a general Lax operator of the form

$$L = \partial^n + \sum_{m=1}^{\infty} A_m \partial^{n-m} \tag{1}$$

where the coefficients, A_m's, are functions of the dynamical variables of the system which depend on the coordinates (x, t). Let us assume that the Lax equation

$$\frac{\partial L}{\partial t_k} = [(L^k)_{\geq s}, L], \qquad s = 0, 1, 2 \tag{2}$$

describes the dynamical system of equations. Here $()_{\geq s}$ represents the projection with respect to the powers of ∂. In going to the dispersionless limit, first of all, we replace $\partial \to p$. Then, we scale $p \to \alpha p$ and all the basic dynamical variables as $J_i \to (\alpha)^i J_i$ where J_i represents the basic dynamical variables of the system with the respective dimensions i. The Lax function which describes the dispersionless system of equations is obtained to be [1]

$$\mathcal{L} = \lim_{\alpha \to \infty} \frac{1}{\alpha^n} L_\alpha \tag{3}$$

where L_α denotes the scaled Lax function. The dispersionless equations are then obtained from the Lax equation

$$\frac{\partial \mathcal{L}}{\partial t_k} = -\left\{ (\mathcal{L}^k)_{\geq s}, \mathcal{L} \right\} \tag{4}$$

where

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} \tag{5}$$

represents the Poisson bracket on the classical phase space.

Let us illustrate this procedure with a few examples. The reduction of the KdV equation to its dispersionless limit is well known and, therefore, we will not repeat it here. Rather, let us look at the two boson hierarchy [8] described by the Lax operator

$$L = \partial - J + \partial^{-1} T \tag{6}$$

with the nonstandard Lax equation given by

$$\frac{\partial L}{\partial t_k} = -\left[(L^k)_{\geq 1}, L\right] \tag{7}$$

Here J and T are dynamical field variables with dimensions one and two respectively. Therefore, under the scaling discussed earlier, we have for the present case, $p \to \alpha p, J \to \alpha J, T \to \alpha^2 T$ and it follows that in the dispersionless limit, the Lax operator goes into

$$\mathcal{L} = p - J + T p^{-1} \tag{8}$$

and the Lax equation

$$\frac{\partial \mathcal{L}}{\partial t_k} = \left\{(\mathcal{L}^k)_{\geq 1}, \mathcal{L}\right\} \tag{9}$$

describes the dispersionless system of equations.

The two boson hierarchy [8] can also be alternatively described in terms of the gauge equivalent Lax operator [9]

$$L = \partial - \frac{1}{\partial + J}\left(\frac{T}{2}\right) \tag{10}$$

and the standard Lax equation

$$\frac{\partial L}{\partial t_k} = \left[\left(L^k\right)_{\geq 0}, L\right] \tag{11}$$

This description of the system of equations is more convenient from the point of view of our subsequent discussions. We note that the second flow

$$J_{t_2} = \left(J_x + T - J^2\right)_x \ , \quad T_{t_2} = -T_{xx} - 2\,(JT)_x \tag{12}$$

is the two boson equation which is also related to the nonlinear Schrödinger equation (NLS) [9]. The third flow, on the other hand, is obtained to be

$$J_{t_3} = J_{xxx} + \left(J^3 - 3JT - 3JJ_x\right)_x \ , \quad T_{t_3} = T_{xxx} + 3\left(J^2 T - \frac{1}{2}T^2 + JT_x\right)_x \tag{13}$$

and this coincides with the bosonic sector of the $N = 2$ supersymmetric KdV equation with $a = 4$ [4] after the transformations

$$J \to 2J \ , \quad T \to -2(T + J_x) \ , \quad x \to ix \ , \quad t \to it \ . \tag{14}$$

In the present case, it is easy to check that, in the dispersionless limit, the Lax function becomes

$$\mathcal{L} = p - \frac{1}{p+J}\left(\frac{T}{2}\right) . \tag{15}$$

The second and the third flow equations following from the Lax equation

$$\frac{\partial \mathcal{L}}{\partial t_k} = -\left\{(\mathcal{L}^k)_{\geq 0}, \mathcal{L}\right\} \tag{16}$$

are given by

$$J_{t_2} = \left(T - J^2\right)_x , \quad T_{t_2} = -2\,(JT)_x \tag{17}$$

$$J_{t_3} = \left(J^3 - 3JT\right)_x , \quad T_{t_3} = 3\left(J^2 T - \frac{1}{2}T^2\right)_x \tag{18}$$

These are indeed the correct dispersionless limits of the two boson hierarchy.

We would like to conclude this section by presenting a new system of dispersionless equations. Let us consider a general Lax operator of the form [6]

$$L = \partial - \frac{1}{\partial^m + \sum_{i=1}^m J_i \partial^{m-i}}\left(\sum_{i=1}^m \overline{J}_i \partial^{m-i}\right) \tag{19}$$

Here $J_i, \overline{J}_i$ represent fields of dimension i. It can be checked that

$$\frac{\partial L}{\partial t_k} = \left[(L^k)_{\geq 0}, L\right] \tag{20}$$

leads to a consistent set of dynamical equations. If we now follow the earlier discussion, it is easy to check that, in the dispersionless limit, the Lax function

$$\mathcal{L} = p - \frac{1}{p^m + \sum_{i=1}^m J_i p^{m-i}}\left(\sum_{i=1}^m \overline{J}_i p^{m-i}\right) \tag{21}$$

leads to the system of dispersionless equations obtained from the Lax equation

$$\frac{\partial \mathcal{L}}{\partial t_k} = -\left\{(\mathcal{L}^k)_{\geq 0}, \mathcal{L}\right\} \tag{22}$$

3 Supersymmetric KdV and two boson hierarchy

In this section, we will generalize the ideas of the previous section to construct the Lax description for a class of dispersionless supersymmetric systems. In particular, we will work out in detail the case of dispersionless $N = 1$ supersymmetric two boson hierarchy starting from the Lax description of the $N = 2$ supersymmetric KdV hierarchy with $a = 4$ [6] whose bosonic sector we have studied in the last section. Let us first note that the few dispersionless supersymmetric systems [2] whose Lax

descriptions have been constructed by brute force show that although the Lax function is defined in terms of superfields, it involves only bosonic momenta and that the conserved charges are obtained from the bosonic residues of powers of the Lax function. Furthermore, in the dispersionless limit, we know that $\partial \to p$. However, the reduction of the fermionic covariant derivative in the dispersionless limit is not well understood. It is also already noted [2] that a scaling of the fermionic covariant derivative is essential in order to preserve supersymmetry in going to the dispersionless limit. In view of the above mentioned difficulties, our strategy, as a first step, is to look at supersymmetric systems which are described in terms of Lax operators that involve only bosonic ∂ operators.

For one of the three known families of integrable supersymmetric hierarchies with $N = 2, W_n$ superalgebra as the second Hamiltonian structure, the Lax operators contain only bosonic operators of the forms [6]:

$$L_s = \partial - \left[D \frac{1}{\partial^s + \sum_{i=1}^{s} J_i \partial^{s-i}} \overline{D} \left(\sum_{i=1}^{s} J_i \partial^{s-i} \right) \right] . \tag{23}$$

Here, $s = 0, 1, 2, \ldots$ and J_i are bosonic $N = 2$ superfields of dimensions i. Furthermore, the square brackets stand for the fact that the $N = 2$ supersymmetric fermionic covariant derivatives D and $\overline{D}$ defined to be

$$D = \frac{\partial}{\partial \theta} - \frac{\overline{\theta}}{2} \partial \, , \quad \overline{D} = \frac{\partial}{\partial \overline{\theta}} - \frac{\theta}{2} \partial \tag{24}$$

act only on the superfields inside the brackets.

Let us consider the conventional dispersionless limit in the simplest case of $N = 2$ supersymmetric KdV hierarchy for which $s = 1$. The Lax operator, in this case, has the form

$$L = \partial - \left[D \frac{1}{\partial + J} \overline{D} J \right] . \tag{25}$$

The second flow, following from this Lax operator, reads

$$J_{t_2} = \left([D, \overline{D}] J - J^2 \right)_x . \tag{26}$$

Under the rescaling $\partial_t \to \lambda \partial_t, \partial \to \lambda \partial, (D, \overline{D}) \to \lambda^{\frac{1}{2}} (D, \overline{D})$, this equation will reduce to (as $\lambda \to 0$)

$$J_{t_2} = - \left(J^2 \right)_x . \tag{27}$$

However, this equation, despite being $N = 2$ supersymmetric, is not very interesting.

A different possibility is to rescale in a standard way all the fields together with the fermionic derivatives in the Lax operator (25). This leads to the following Lax function in the dispersionless limit:

$$\mathcal{L} = p - \frac{\frac{1}{2} \mathcal{T}}{p + \mathcal{J}} - \frac{\frac{1}{2} \psi_1 \psi_2}{(p + \mathcal{J})^2} , \tag{28}$$

where we introduced the component fields:

$$\mathcal{J} = J| \, , \; \psi_1 = (D + \overline{D}) J| \, , \; \psi_2 = (D - \overline{D}) J| \, , \; \mathcal{T} = \left[D, \overline{D} \right] J| \, , \tag{29}$$

and the restriction | stands for keeping the $(\theta=\bar{\theta}=0)$ term. One can check, that the second flow equations

$$
\begin{aligned}
&\mathcal{J}_{t_2}=\left(\mathcal{T}-\mathcal{J}^2\right)_x\ , \quad \mathcal{T}_{t_2}=-2\left(\mathcal{J}\mathcal{T}-\psi_1\psi_2\right)_x\ ,\\
&(\psi_1)_{t_2}=-2\left(\mathcal{J}\psi_1\right)_x\ , \quad (\psi_2)_{t_2}=-2\left(\mathcal{J}\psi_2\right)_x\ ,
\end{aligned}
\tag{30}
$$

do not possess any supersymmetry at all. They break even the $N=1$ supersymmetry. The same is also true for higher flows.

Therefore we propose the following alternative approach to taking the dispersionless limit. The main idea is to rescale the fermionic components of the superfields J_i differently from the conventional method as

$$
\begin{aligned}
&J_i| \rightarrow \alpha^i\ J_i|\ ,\ \left(D+\overline{D}\right)J_i\Big| \rightarrow \alpha^i\left(D+\overline{D}\right)J_i\Big|\ ,\\
&\left(D-\overline{D}\right)J_i\Big| \rightarrow \alpha^{i+1}\left(D-\overline{D}\right)J_i\Big|\ ,\ \left[D,\overline{D}\right]J_i\Big| \rightarrow \alpha^{i+1}\left[D,\overline{D}\right]J_i\Big|\ .
\end{aligned}
\tag{31}
$$

It is clear that this unconventional, alternative rescaling (31) will explicitly break the $N=2$ supersymmetry. However, a subset of $N=1$ supersymmetry, generated by $(D+\overline{D})$ will survive and, in the dispersionless limit, we will have a Lax description for an $N=1$ supersymmetric system of equations.

Let us demonstrate in detail how all these work for the Lax operator (25). According to our alternative procedure, the first step will consist of representing $\frac{1}{\partial+J}$ as

$$
\frac{1}{\partial+J}\equiv\partial^{-1}+A_2\partial^{-2}+A_3\partial^{-3}+A_4\partial^{-4}+\dots\ , \tag{32}
$$

where all the functions A_n can be recursively calculated and the first few have the explicit forms

$$
A_2=-J\ ,\ A_3=J^2+J_x\ ,\ A_4=-J^3-3JJ_x-J_{xx}\ ,\ \dots\ . \tag{33}
$$

Thus, our Lax operator can also be written as

$$
L=\partial-\left[D\left(\partial^{-1}\overline{D}J+A_2\partial^{-2}\overline{D}J+A_3\partial^{-3}\overline{D}J+A_4\partial^{-4}\overline{D}J+\dots\right)\right]\ , \tag{34}
$$

and we should move the partial derivatives to the right in (34).

The first non trivial term on the right hand side of (34) generates an infinite series of terms when the derivative is moved to the right, namely,

$$
\partial^{-1}\left[D\overline{D}J\right]\equiv\frac{1}{2}\left(\mathcal{T}-\mathcal{J}_x\right)\partial^{-1}-\frac{1}{2}\left(\mathcal{T}-\mathcal{J}_x\right)_x\partial^{-2}+\frac{1}{2}\left(\mathcal{T}-\mathcal{J}_x\right)_{xx}\partial^{-3}+\dots\ . \tag{35}
$$

We may now replace $\partial\rightarrow p$ in the r.h.s. of (35) and rescale

$$
p\rightarrow\alpha p\ ,\ \mathcal{J}\rightarrow\alpha\mathcal{J}\ ,\ \psi_1\rightarrow\alpha\psi_1\ ,\ \psi_2\rightarrow\alpha^2\psi_2\ ,\ \mathcal{T}\rightarrow\alpha^2\mathcal{T}\ .
$$

Then, it is easy to see that the only term from among those in (35) that will contribute to $\lim_{\alpha\to\infty}\frac{1}{\alpha}L_\alpha$ is

$$\frac{1}{2}\mathcal{T}p^{-1}.$$

The second term inside the square bracket in the right hand side of (34) needs some more work:

$$(DA_2)\partial^{-2}(\overline{D}J) + A_2\partial^{-2}(D\overline{D}J) \equiv (DA_2)(\overline{D}J)\partial^{-2} - 2(DA_2)(\overline{D}J)_x\partial^{-3} + A_2(D\overline{D}J)\partial^{-2} - 2A_2(D\overline{D}J)_x\partial^{-3} + \dots\ , \tag{36}$$

where the dots stand for terms with ∂^{-4} and higher. In the scaling limit, only the following terms will survive

$$\frac{1}{2}\psi_1\psi_2 p^{-2} - \frac{1}{2}\psi_2\left(\psi_2\right)_x p^{-3} - \frac{1}{2}\mathcal{J}\mathcal{T}p^{-2}\ . \tag{37}$$

Continuing in a similar manner, we find the Lax operator in the dispersionless limit to be

$$\mathcal{L} = p - \frac{\frac{1}{2}\mathcal{T}}{p+\mathcal{J}} - \frac{\frac{1}{2}\psi_1\psi_2}{(p+\mathcal{J})^2} + \frac{\frac{1}{4}\psi_2(\psi_2)_x}{(p+\mathcal{J})^3}\ . \tag{38}$$

The second flow, following from the Lax equation, has the form

$$\begin{aligned}&\mathcal{J}_{t_2} = \left(\mathcal{T} - \mathcal{J}^2\right)_x\ , \quad \mathcal{T}_{t_2} = -2\left(\mathcal{J}\mathcal{T} - \psi_1\psi_2\right)_x\ ,\\ &\left(\psi_1\right)_{t_2} = \left((\psi_2)_x - 2\mathcal{J}\psi_1\right)_x\ , \quad \left(\psi_2\right)_{t_2} = -2\left(\mathcal{J}\psi_2\right)_x\ .\end{aligned} \tag{39}$$

These equations can also be easily rewritten in terms of $N=1$ superfields,

$$j = \mathcal{J} + \theta\psi_1\ , \quad \psi = \psi_2 - \theta\mathcal{T} \tag{40}$$

as

$$j_{t_2} = -\left(\mathcal{D}\psi + j^2\right)_x\ , \quad \psi_{t_2} = -2\left(j\psi\right)_x\ , \tag{41}$$

where

$$\mathcal{D} = \frac{\partial}{\partial\theta} - \theta\partial\ , \quad \mathcal{D}^2 = -\partial\ .$$

Let us note here that the Lax operator (38) is not new and is gauge equivalent to the one which has been constructed earlier by brute force in [3]. However, we see that it can be systematically obtained from the alternative dispersionless limit of the simplest of the Lax operators in the family (23).

4 Supersymmetric Boussinesq hierarchy

As a second example of our method, in this section, we will work out the dispersionless limit starting from the $N=2$ supersymmetric Boussinesq hierarchy with $\alpha = \frac{5}{2}$ [7], which is described by the Lax operator (23) with $s=2$ [6]. The Lax operator (23), in this case, has the explicit form

$$L = \partial - \left[D\frac{1}{\partial^2 + J_1\partial + J_2}\overline{D}\left(J_1\partial + J_2\right)\right]\ . \tag{42}$$

Following our procedure, we will first rewrite

$$\frac{1}{\partial^2 + J_1\partial + J_2} = \partial^{-2} + A_1\partial^{-3} + A_2\partial^{-4} + \dots \ , \tag{43}$$

where all the A_n's can be easily calculated,

$$A_1 = -J_1 \ , \ A_2 = -J_2 + 2(J_1)_x + J_1^2 \ , \ A_3 = \left(2J_2 - 3(J_1)_x - \frac{5}{2}J_1^2\right)_x + 2J_1J_2 - J_1^3 \ , \ \cdots \tag{44}$$

With this, the first few terms in the square bracket in (42) have the form

$$\begin{aligned}\left[D\frac{1}{\partial^2 + J_1\partial + J_2}\overline{D}\,(J_1\partial + J_2)\right] &= \left[D\partial^{-2}\overline{D}\,(J_1\partial + J_2)\right] + \left[DA_1\partial^{-3}\overline{D}\,(J_1\partial + J_2)\right] + \\ &\quad \left[DA_2\partial^{-4}\overline{D}\,(J_1\partial + J_2)\right] + \left[DA_3\partial^{-5}\overline{D}\,(J_1\partial + J_2)\right] + \dots\end{aligned} \tag{45}$$

Let us next introduce the components

$$\begin{aligned}&\mathcal{J}_1 = J_1| \ , \ \psi_1 = (D+\overline{D})J_1| \ , \ \psi_2 = (D-\overline{D})J_1| \ , \ \mathcal{T}_1 = \left[D,\overline{D}\right] J_1| \ , \\ &\mathcal{T}_2 = J_2| \ , \ \xi_1 = (D+\overline{D})J_2| \ , \ \xi_2 = (D-\overline{D})J_2| \ , \ \mathcal{W} = \left[D,\overline{D}\right] J_2| \ ,\end{aligned} \tag{46}$$

which have the scaling behaviors:

$$(\mathcal{J}_1,\psi_1) \to \alpha\,(\mathcal{J}_1,\psi_1) \ , \ (\psi_2,\mathcal{T}_1,\mathcal{T}_2,\xi_1) \to \alpha^2\,(\psi_2,\mathcal{T}_1,\mathcal{T}_2,\xi_1) \ , \ (\xi_2,\mathcal{W}) \to \alpha^3\,(\xi_2,\mathcal{W})\,. \tag{47}$$

We are now ready to find a Lax function in the dispersionless limit.

We can now have the fermionic derivatives act on the fields in (45), move the partial derivatives to the right and replace $\partial \to p$. After this, it is easy to see that there will be three types of terms that may survive in the limit (3):

$$\mathcal{L} \equiv p - \mathcal{A} - \mathcal{B} - \mathcal{C} \ ,$$

$$\mathcal{A} \equiv \frac{1}{2}\,(\mathcal{T}_1 p + \mathcal{W})\left(p^{-2} + A_1 p^{-3} + A_2 p^{-4} + \dots\right) \tag{48}$$

$$\mathcal{B} \equiv \left((DA_1)p^{-3} + (DA_2)p^{-4} + \dots\right)\left((\overline{D}J_1)p + (\overline{D}J_2)\right) \tag{49}$$

$$\mathcal{C} \equiv \left(-3(DA_1)p^{-4} - 4(DA_2)p^{-5} + \dots\right)\left((\overline{D}J_1)p + (\overline{D}J_2)\right)_x \ . \tag{50}$$

Note that the expressions in the parenthesis for $\mathcal{A},\mathcal{B},\mathcal{C}$ contain terms with and without derivatives (see eq.(44)). For terms of the types $\mathcal{A}$ and $\mathcal{C}$, there is no problem, since in the dispersionless limit (scaling limit), only terms without derivatives in A_n (44) contribute. In this case, we have:

$$\mathcal{A} = \frac{\frac{1}{2}\,(\mathcal{T}_1 p + \mathcal{W})}{p^2 + \mathcal{J}_1 p + \mathcal{T}_2} \ , \quad \mathcal{C} = -\left[D\frac{2p + J_1}{\left(p^2 + J_1 p + J_2\right)^2}\right]\left((\overline{D}J_1)p + (\overline{D}J_2)\right)_x \ . \tag{51}$$

However, for terms of the type $\mathcal{B}$, the scaling require us to keep also the terms linear in the first derivatives in all the A_n's. This leads to

$$\mathcal{B} = \left[D \left(\frac{1}{p^2 + J_1 p + J_2} + \frac{(2p + J_1)(J_1 p + J_2)_x}{(p^2 + J_1 p + J_2)^3} \right) \right] \left((\overline{D} J_1) p + (\overline{D} J_2) \right) \ . \tag{52}$$

In the dispersionless limit, the Lax function now becomes

$$\begin{aligned} \mathcal{L} \ &= \ p - \frac{\frac{1}{2}\left(\mathcal{T}_1 p + \mathcal{W}\right)}{p^2 + \mathcal{J}_1 p + \mathcal{T}_2} - \frac{\frac{1}{4}\psi_2 \left(\psi_2 p + \xi_2\right)_x}{\left(p^2 + \mathcal{J}_1 p + \mathcal{T}_2\right)^2} - \frac{\frac{1}{2}\left(\psi_1 p + \xi_1\right)\left(\psi_2 p + \xi_2\right)}{\left(p^2 + \mathcal{J}_1 p + \mathcal{T}_2\right)^2} + \\ &\quad \frac{\frac{1}{4}(2p + \mathcal{J}_1)\left(\psi_2 p + \xi_2\right)\left(\psi_2 p + \xi_2\right)_x}{\left(p^2 + \mathcal{J}_1 p + \mathcal{T}_2\right)^3} + \frac{\frac{1}{4}\left(\mathcal{J}_1 p + \mathcal{T}_2\right)_x \psi_2 \xi_2}{\left(p^2 + \mathcal{J}_1 p + \mathcal{T}_2\right)^3} \ . \end{aligned} \tag{53}$$

It is now easy to check that the Lax equation (22) leads to the dispersionless supersymmetric Boussinesq hierarchy. Explicitly, the second flow of this hierarchy is given by

$$\begin{aligned} &(\mathcal{J}_1)_{t_2} = \left(2\mathcal{T}_1 + 2\mathcal{T}_2 - \mathcal{J}_1^2\right)_x \ , \\ &(\mathcal{T}_1)_{t_2} = 2\left(\mathcal{W} - \mathcal{J}_1\mathcal{T}_1 + \psi_1\psi_2\right)_x \ , \ (\mathcal{T}_2)_{t_2} = -2(\mathcal{J}_1)_x\mathcal{T}_2 + \mathcal{J}_1(\mathcal{T}_1)_x \ , \\ &(\mathcal{W})_{t_2} = -2(\mathcal{J}_1)_x\mathcal{W} - 2(\mathcal{T}_1)_x\mathcal{T}_2 + \mathcal{T}_1(\mathcal{T}_1)_x + \psi_2(\psi_2)_{xx} + 2\left(\xi_1(\psi_2)_x - \xi_2(\psi_1)_x\right) \ , \\ &(\psi_1)_{t_2} = 2\left(\xi_1 + (\psi_2)_x - \mathcal{J}_1\psi_1\right)_x \ , \ (\psi_2)_{t_2} = 2\left(\xi_2 - \mathcal{J}_1\psi_2\right)_x \ , \\ &(\xi_1)_{t_2} = -2(\mathcal{J}_1)_x\xi_1 - 2\mathcal{T}_2(\psi_1)_x + (\mathcal{T}_1)_x\psi_1 + \mathcal{J}_1(\psi_2)_{xx} \ , \\ &(\xi_2)_{t_2} = -2(\mathcal{J}_1)\xi_2 - 2\mathcal{T}_2(\psi_2)_x + (\mathcal{T}_1)_x\psi_2 \ . \end{aligned} \tag{54}$$

This system of equations can also be rewritten in terms of $N = 1$ superfields

$$j_1 = \mathcal{J}_1 + \theta\psi_1 \ , \ \eta_1 = \psi_2 - \theta\mathcal{T}_1 \ , \ j_2 = \mathcal{T}_2 + \theta\xi_1 \ , \ \eta_2 = \xi_2 - \theta\mathcal{W} \tag{55}$$

as

$$\begin{aligned} &(j_1)_{t_2} = \left(-2\mathcal{D}\eta_1 + 2j_2 - j_1^2\right)_x \qquad (\eta_1)_{t_2} = 2\left(\eta_2 - j_1\eta_1\right)_x \ , \\ &(j_2)_{t_2} = -2(j_1)_x j_2 - j_1(\mathcal{D}\eta_1)_x \ , \ (\eta_2)_{t_2} = -2(j_1)_x\eta_2 - 2j_2(\eta_1)_x - (\mathcal{D}\eta_1)_x\eta_1 \ . \end{aligned} \tag{56}$$

Thus, we explicitly demonstrate that our system possesses $N = 1$ supersymmetry.

5 Conclusion

In this paper, we have given a systematic derivation of the dispersionless limit of a class of $N = 1$ supersymmetric models starting from the Lax description of their dispersive counterparts. This is achieved by starting with the $N = 2$ systems and making an alternative scaling of the field variables which maintains only an $N = 1$ supersymmetry. This approach is motivated by the structure of the dispersionless limit of the pure bosonic sectors of these $N = 2$ systems which can not be extended to $N = 2$ supersymmetry without introducing the additional bosonic fields. We discuss our proposal explicitly within the context of the supersymmetric two boson hierarchy where our starting point is the $N = 2$ supersymmetric KdV hierarchy. As a second example, we also work out the dispersionless limit of the supersymmetric Boussinesq hierarchy starting from the $N = 2$ supersymmetric system.

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